

Source Compression with a Quantum Helper

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Abstract—We study classical source coding with quantum side-information where the quantum side-information is observed by a helper and sent to the decoder via a classical channel. We derive a single-letter characterization of the achievable rate region for this problem. The direct part of our result is proved via the measurement compression theory by Winter. Our result reveals that a helper's scheme that separately conducts a measurement and a compression is suboptimal, and the measurement compression is fundamentally needed to achieve the optimal rate region.

I. INTRODUCTION

Source coding normally refers to the information processing task that aims to reduce the redundancy exhibited when multiple copies of the same source are used. In establishing information theory, Shannon demonstrated a fundamental result that source coding can be done in a *lossless* fashion; namely, the recovered source will be an exact replica of the original one when the number of copies of the source goes to infinity [1]. If representing the source by a random variable X with output space \mathcal{X} and distribution p_X , lossless source coding is possible if and only if the compression rate R is above its Shannon entropy:

$$R \geq H(X), \quad (1)$$

where $H(X) := \sum_{x \in \mathcal{X}} -p_X(x) \log p_X(x)$.

Redundancy can also exist in the scenario in which multiple copies of the source are shared by two or more parties that are far apart. Compression in this particular setting is called *distributed* source coding, which has been proven to be extremely important in the internet era. The goal is to minimise the information sent by each party so that the decoder can still recover the source faithfully. Shannon's lossless source coding theorem can still be applied individually to each party. However, it is discovered that a better source coding strategy exists if the sources between different parties are correlated. Denote X and Y the sources held by the two distant parties, where the joint distribution is P_{XY} and the output spaces are \mathcal{X} and \mathcal{Y} , respectively. Slepian and Wolf showed that lossless distributed source coding is possible when the compression rates R_1 and R_2 for the two parties satisfy [2]:

$$R_1 \geq H(X|Y), \quad (2)$$

$$R_2 \geq H(Y|X), \quad (3)$$

$$R_1 + R_2 \geq H(XY), \quad (4)$$

where $H(X|Y)$ is the conditional Shannon entropy. This theorem is now called the classical Slepian-Wolf theorem [2]. In particular, when source Y is directly observed at the

decoder, the problem is sometimes called source coding with (full) side-information.

Another commonly encountered scenario in a communication network is that a centralised server exists and its role is to coordinate all the information processing tasks, including the task of source coding, between the nodes in this network. Obviously, the role of the server is simply as a helper and it is not critical to reproduce the exact information communicated by the server. This slightly different scenario results in a completely different characterisation of the rate region, as observed by Wyner [3] and Ahlswede-Körner [4]. Consider that the receiver wants to recover the source X with the assistance of the server (that we will call a helper from now on) holding Y , where the distribution is P_{XY} . Wyner showed that the optimal rate region for lossless source coding of X with a classical helper Y is the set of rate pairs (R_1, R_2) such that

$$R_1 \geq H(X|U), \quad (5)$$

$$R_2 \geq I(U; Y), \quad (6)$$

for some conditional distribution $p_{U|Y}(u|y)$, and $I(U; Y)$ is the classical mutual information between random variables U and Y . When there is no constraint on R_2 (i.e. R_2 can be as large as it can be), this problem reduces to source coding with (full) side-information.

The problem of source coding, when replacing classical sources with quantum sources, appears to be highly nontrivial in the first place¹. The first quantum source coding theorem was established by Schumacher [5], [6]. A quantum source ρ_A can be losslessly compressed and decompressed if and only if the rate R is above its von Neumann entropy²:

$$R \geq H(A)_\rho, \quad (7)$$

where $H(A)_\rho := -\text{Tr } \rho_A \log \rho_A$.

Schumacher's quantum source coding theorem bears a close resemblance to its classical counterpart. One will naturally expect that the same will hold true for the distributed source coding problem in the quantum regime. Consider that Alice, who has the quantum system A of an entangled source ρ_{AB} , would like to merge her state to the distant party Bob. Then, the rate R at which quantum states with density matrix ρ_A can be communicated to a party with quantum side information ρ_B

¹The quantum source coding result takes a much longer time to develop if one considers that quantum theory began to evolve in the mid-1920s.

²The subscript A is a label to which the quantum system ρ_A belongs.

is given by the conditional von Neumann entropy $H(A|B)_\rho$, a simple observation followed from the classical Slepian-Wolf theorem. While this naive conclusion turns out to be correct, this result has a much deeper and profound impact in the theory of quantum information as it marks a clear departure between classical and quantum information theory. It is rather perplexing that the rate R is quantified by the conditional entropy $H(A|B)_\rho$, which can be negative. This major piece of the puzzle was resolved with the interpretation that if the rate is negative, the state can be merged, and in addition, the two parties will gain $|H(A|B)_\rho|$ amount of entanglement for later quantum communication [8], [9], [10]. The distributed quantum source coding problem was later fully solved [11], [12] where the trade-off rate region between the quantum communication and the entanglement resource is derived. The result is now called the fully quantum Slepian-Wolf theorem (FQSW).

Source coding with hybrid classical-quantum systems ρ_{XB} with X representing a classical system and B a quantum state is also considered in quantum information theory, and our result falls into this category. In [13], Devetak and Winter considered classical source coding with quantum side information at the decoder, and showed that the optimal rate R_1 is given by $H(X|B)_\rho$. This result can be regarded as a classical-quantum version of the source coding with (full) side-information.

In this work, we consider classical source coding with a quantum helper, a problem that was completely overlooked before. In our problem, the quantum side-information is observed by the helper, and the decoder will only have a classical description from the quantum helper. Although our problem can be regarded as a classical-quantum version of the classical helper problem studied in [3], [4], in contrast to its classical counterpart, our problem does not reduce to source coding with quantum side-information studied in [13] even if there is no constraint on rate R_2 . However, when the ensemble that constitutes the quantum side-information is commutative, our problem reduces to the classical helper problem.

We completely characterize the rate region of the quantum helper problem. In fact, the formulae describing the rate region (cf. Theorem 1) resembles its classical counterpart (cf. (5) and (6)). However, the proof technique is very different due to the quantum nature of the helper. In particular, we use the measurement compression theory by Winter [22] in the direct coding theorem. One of interesting consequences of our result is that a helper's scheme that separately conducts a measurement and a compression is suboptimal; measurement compression is fundamentally needed to achieve the optimal rate region.

There are a huge amount of work devoted to both classical and quantum lossy source coding [14], [15], [16], [17], [18], [19]. However, we will restrict ourselves to only noiseless source coding in this work.

Notations. In this paper, we will use capital letters X, Y, Z, U etc. to denote classical random variables, and lower cases x, y, z, u to denote their realisations. We use $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}$ to denote the sample spaces. We denote $x^n = x_1 x_2 \cdots x_n$.

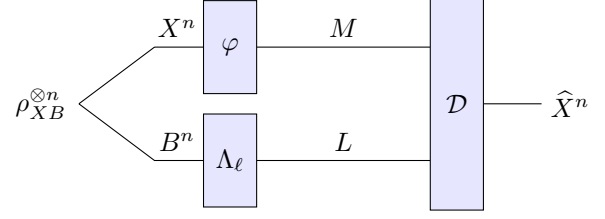


Fig. 1. Source Compression with a Quantum Helper.

A quantum state is a positive semi-definite matrix with trace equal to one. We will use ρ or σ to denote a quantum system in this paper. In case we need to specify which party the quantum state belongs to, we will use a subscript description ρ_A , meaning that the quantum system is held by A(lice). Letting $\{|x\rangle\langle x|\}_{x \in \mathcal{X}}$ be a set of orthonormal basis vectors, a classical-quantum state ρ_{XB} is written as

$$\rho_{XB} = \sum_x p_X(x) |x\rangle\langle x| \otimes \rho_x,$$

so that n copies of it is

$$\rho_{XB}^{\otimes n} = \sum_{x^n} p_X^{(n)}(x^n) |x^n\rangle\langle x^n| \otimes \rho_{x^n},$$

where we denote $\rho_{x^n} := \rho_{x_1} \otimes \cdots \otimes \rho_{x_n}$ for the sequence x^n . A positive-operator valued measure (POVM), $\Lambda = \{\Lambda_y\}$, is a quantum measurement whose elements are non-negative self-adjoint operators on a Hilbert space so that $\sum_{y \in \mathcal{Y}} \Lambda_y = I$.

This paper is organised as follows. In Sec II, we formally define the problem of source coding with a quantum helper, and present the main result as well as its proof. We conclude in Sec III with open questions.

II. CLASSICAL SOURCE COMPRESSION WITH A QUANTUM HELPER

As shown in Figure 1, the protocol for classical source coding with a quantum helper involves two senders, Alice and Bob, and one receiver, Charlie. Initially Alice and Bob hold n copies of a classical-quantum state ρ_{XB} . In this case, Alice holds classical random variables X^n while Bob (being a helper) holds a quantum state ρ_{X^n} that is correlated with Alice's message. The goal is for the decoder Charlie to faithfully recover Alice's message when assisted by the quantum helper Bob.

We now proceed to formally define the coding procedure. We define an (n, ϵ) code for classical source compression with a quantum helper to consist of the following:

- Alice's encoding operation $\varphi : \mathcal{X}^n \rightarrow \mathcal{M}$, where $\mathcal{M} := \{1, 2, \dots, |\mathcal{M}|\}$ and $|\mathcal{M}| = 2^{nR_1}$;
- Bob's POVM $\Lambda = \{\Lambda_\ell\} : \mathcal{B}^n \rightarrow \mathcal{L}$, where $\mathcal{L} := \{1, 2, \dots, |\mathcal{L}|\}$ and $|\mathcal{L}| = 2^{nR_2}$;
- Charlie's decoding operation $\mathcal{D} : \mathcal{M} \times \mathcal{L} \rightarrow \hat{\mathcal{X}}^n$

so that the error probability satisfies

$$\Pr\{X^n \neq \hat{X}^n\} \leq \epsilon. \quad (8)$$

A rate pair (R_1, R_2) is said to be *achievable* if for any $\epsilon, \delta > 0$ and all sufficiently large n , there exists an (n, ϵ) code with rates $R_1 + \delta$ and $R_2 + \delta$. The rate region is then defined as the collection of all achievable rate pairs. Our main result is the following theorem.

Theorem 1: Given is a classical-quantum source ρ_{XB} . The optimal rate region for lossless source coding of X with a quantum helper B is the set of rate pairs (R_1, R_2) such that

$$R_1 \geq H(X|U) \quad (9)$$

$$R_2 \geq I(U; B)_\sigma. \quad (10)$$

The state $\sigma_{UB}(\Lambda)$ resulting from Bob's application of the POVM $\Lambda = \{\Lambda_u\}_{u \in \mathcal{U}}$ is

$$\sigma_{UB}(\Lambda) = \sum_{u \in \mathcal{U}} p_U(u) |u\rangle\langle u| \otimes \rho_u \quad (11)$$

where

$$p_U(u) = \text{Tr}(\rho_B \Lambda_u) \quad (12)$$

$$\rho_u = \frac{1}{p_U(u)} [\sqrt{\rho_B} \Lambda_u \sqrt{\rho_B}]^* \quad (13)$$

$$\rho_B = \sum_x p_X(x) \rho_x. \quad (14)$$

where $*$ denotes complex conjugation in the standard basis. Furthermore, we can restrict the size of POVM as $|\mathcal{U}| \leq d_B^2$, where d_B is the dimension of Bob's system.

A typical shape of the rate region in Theorem 1 is described in Fig. 2. When there is no constraint on R_2 , rate R_1 can be decreased as small as

$$H(X|U^*) := \min_{\Lambda} H(X|U) \quad (15)$$

$$= H(X) - \max_{\Lambda} I(X; U) \quad (16)$$

$$= H(X) - I_{\text{acc}}, \quad (17)$$

where I_{acc} is the accessible information for the ensemble $\{(p_X(x), \rho_x)\}_{x \in \mathcal{X}}$. Unless the ensemble commutes [7], the minimum rate $H(X|U^*)$ is larger than the rate $H(X|B)_\rho$, which is the optimal rate in the source coding with quantum side-information [13]. To achieve $R_1 = H(X|U^*)$, it suffices to have $R_2 \geq I(U^*; B)_\sigma$, which is smaller than $H(U^*)$ in general. This means that the following separation scheme is suboptimal: first conduct a measurement to get U^* and then compress U^* . For more detail, see the direct coding proof.

Converse:

Let $\varphi : \mathcal{X}^n \rightarrow \mathcal{M}$ be Alice's encoder, and let $\{\Lambda_\ell\}_{\ell \in \mathcal{L}}$ be Bob's measurement. Alice sends $M = \varphi(X^n)$ to the decoder, and Bob sends the measurement outcome L to the decoder. The Fano's inequality states that $H(X^n|M, L) \leq n\epsilon_n$ for some $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$.

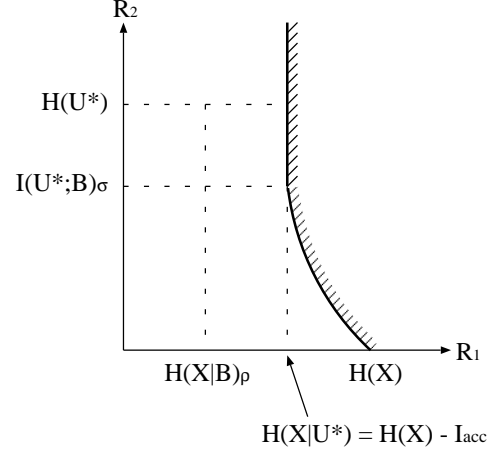


Fig. 2. A typical shape of the rate region in Theorem 1.

First, we have the following bound:

$$\log |\mathcal{M}| \geq H(M) \quad (18)$$

$$\geq H(M|L) \quad (19)$$

$$\geq H(X^n|L) - H(X^n|M, L) \quad (20)$$

$$\stackrel{(a)}{\geq} H(X^n|L) - n\epsilon_n \quad (21)$$

$$\stackrel{(b)}{\geq} \sum_{t=1}^n H(X_t|X_{<t}, L) - n\epsilon_n \quad (22)$$

$$\stackrel{(c)}{=} \sum_{t=1}^n H(X_t|U_t) - n\epsilon_n \quad (23)$$

$$\stackrel{(d)}{=} nH(X_J|U_J, J) - n\epsilon_n, \quad (24)$$

where (a) follows from Fano's inequality: $H(X^n|M, L) \leq n\epsilon_n$ for some $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$; in (b), we use chain rule and denote $X_{<t} := (X_1, \dots, X_{t-1})$; in (c), we denote $U_t := (X_{<t}, L)$; in (d), we introduce a time-sharing random variable J that is uniformly distributed in the set $\{1, 2, \dots, n\}$.

Next, we have

$$\log |\mathcal{L}| \geq H(L) \quad (25)$$

$$\geq I(L; B^n) \quad (26)$$

$$= \sum_{t=1}^n I(L; B_t|B_{<t}) \quad (27)$$

$$= \sum_{t=1}^n I(L, B_{<t}; B_t) \quad (28)$$

$$\stackrel{(a)}{=} \sum_{t=1}^n I(L, B_{<t}, X_{<t}; B_t) \quad (29)$$

$$\geq \sum_{t=1}^n I(L, X_{<t}; B_t) \quad (30)$$

$$= \sum_{t=1}^n I(U_t; B_t) \quad (31)$$

where (a) follows from

$$I(X_{<t}; B_t | L, B_{<t}) \leq I(X_{<t}; B_t, B_{>t} | L, B_{<t}) \quad (32)$$

$$= H(X_{<t} | L, B_{<t}) - H(X_{<t} | L, B^n) \quad (33)$$

$$\leq H(X_{<t} | B_{<t}) - H(X_{<t} | L, B^n) \quad (34)$$

$$= H(X_{<t} | B_{<t}) - H(X_{<t} | B^n) \quad (35)$$

$$= H(X_{<t} | B_{<t}) - H(X_{<t} | B_{<t}) \quad (36)$$

$$= 0. \quad (37)$$

Following from Eq. (31), we can again introduce a time-sharing random variable J that is uniformly distributed in the set $\{1, 2, \dots, n\}$,

$$\sum_{t=1}^n I(U_t; B_t) = n \sum_{t=1}^n I(U_t; B_t | J = t) \quad (38)$$

$$= nI(U_J; B_J | J) \quad (39)$$

$$= nI(U_J J; B_J) \quad (40)$$

where the last equality follows because $I(J; B_J) = 0$. To get single-letter formula, define $X = X_J$, $B = B_J$, and $U = (U_J, J)$ and let $n \rightarrow \infty$:

$$R_1 = \frac{1}{n} \log |\mathcal{M}| \geq H(X|U) \quad (41)$$

$$R_2 = \frac{1}{n} \log |\mathcal{L}| \geq I(U; B). \quad (42)$$

Here, we note that the distribution of $U_t = (L, X_{<t})$ can be written as

$$p_{X_{<t}L}(x_{<t}, \ell) = \left(\prod_{i<t} p_{X_i}(x_i) \right) \times \text{Tr} \left[\left\{ \left(\bigotimes_{i<t} \rho_{x_i} \right) \otimes \rho_{B_t} \otimes \left(\bigotimes_{i>t} \rho_{B_i} \right) \right\} \Lambda_\ell \right]. \quad (43)$$

Thus, we can get U_t as a measurement outcome of B_t by first generating $X_{<t}$, then by appending $\bigotimes_{i<t} \rho_{x_i}$ and $\bigotimes_{i>t} \rho_{B_i}$ to ancillae systems, and finally by conducting the measurement $\{\Lambda_\ell\}_{\ell \in \mathcal{L}}$.

Finally, the bound on $|\mathcal{U}|$ can be proved via Carathéodory's theorem (cf. [25, Appendix C]).

Direct Coding Theorem: Fix a POVM measurement $\Lambda = \{\Lambda_u\}_{u \in \mathcal{U}}$. It induces a conditional probability $p_{U|X}(u|x) = \text{Tr}[\Lambda_u \rho_x]$, and joint probability distribution

$$P_{XU}(x, u) = p_X(x) p_{U|X}(u|x). \quad (44)$$

The crucial observation is the application of Winter's measurement compression theory [22].

Theorem 2 (Measurement compression theorem [22], [23]): Let ρ_A be a source state and Λ a POVM to simulate on this state. A protocol for a faithful simulation of the POVM is achievable with classical communication rate R and common randomness rate S if and only if the following set of inequalities hold

$$R \geq I(X; R), \quad R + S \geq H(X), \quad (45)$$

where the entropies are with respect to a state of the following form:

$$\sum_x |x\rangle \langle x|^X \otimes \text{Tr}_A \{ (I^R \otimes \Lambda_x^A) \phi^{RA} \}, \quad (46)$$

and ϕ^{RA} is some purification of the state ρ_A .

Let K be a random variable on \mathcal{K} , which describes the common randomness shared between Alice and Bob. Let $\{\tilde{\Lambda}_{u^n}^{(k)}\}_{u^n \in \mathcal{U}^n}$ be collection of POVMs. Let

$$Q_{X\tilde{U}}^n(x^n, u^n) := P_X^{(n)}(x^n) \sum_{k \in \mathcal{K}} \frac{1}{|\mathcal{K}|} \text{Tr}[\rho_{x^n} \tilde{\Lambda}_{u^n}^{(k)}], \quad (47)$$

where $P_X^{(n)}(x^n) := P_X(x_1) \times \dots \times P_X(x_n)$. The faithful simulation of n copies of POVM $\Lambda := \{\Lambda_u\}_{u \in \mathcal{U}}$, i.e. $\Lambda^{\otimes n}$, implies that for any $\epsilon > 0$, there exists n sufficiently large, such that there exist POVMs $\{\tilde{\Lambda}^{(k)}\}$, where $\tilde{\Lambda}^{(k)} := \{\tilde{\Lambda}_{u^n}^{(k)}\}_{u^n \in \mathcal{U}^n}$, with

$$\frac{1}{2} \|P_{XU}^{(n)} - Q_{X\tilde{U}}^n\|_1 \leq \epsilon. \quad (48)$$

Coding Strategy:

Alice and Bob shared n copies of the state ρ_{XB} , and assume that Bob performs measurement $\Lambda^{\otimes n} : B^{\otimes n} \rightarrow \mathcal{U}^n$ on his quantum system whose outcome is sent to the decoder to assist decoding Alice's message. Bob's measurement on each copy of ρ_{XB} will induce the probability distribution P_{XU} according to (44). Apparently, if Bob sends the full measurement outcomes to Charlie (say $nH(U)$ bits), then Charlie can successfully decode X^n simply from Slepian-Wolf Theorem. The next strategy is to make use of classical result since after Bob's measurement, Alice and Bob become fully classical with joint distribution P_{XU} . Therefore, the minimum rate for Bob is $I(V; U)$ (w.r.t. some conditional distribution $p_{V|U}(v|u)$). However, there is a non-trivial quantum coding strategy. Detail follows.

Bob's coding. Instead of the measurement Λ performed on Bob's system ρ_B and coding w.r.t. the classical channel $p_{V|U}(v|u)$, the decoder Charlie can directly simulate the measurement outcome U using Winter's measurement compression theorem [22], [23]. Denote Bob's classical communication rate $R_2 = \frac{1}{n} \max_{k \in \mathcal{K}} |\tilde{\Lambda}^{(k)}|$. Then Theorem 2 promises that by sending $R_2 \geq I(U; B)$ from Bob to the decoder Charlie, Charlie will have a local copy \tilde{U}^n and the distribution between Alice and Charlie $Q_{X\tilde{U}}^n$ will satisfy (48).

Alice's coding. Now Alice's strategy is very simple since Charlie has had \tilde{U}^n . She just uses the Slepian-Wolf coding strategy as if she starts with the distribution P_{XU} with Charlie. In fact, it is well known (cf. [20]) that there exists an encoder $\varphi : \mathcal{X}^n \rightarrow \mathcal{M}$ and a decoder $\mathcal{D} : \mathcal{M} \times \mathcal{U}^n \rightarrow \mathcal{X}^n$ such that $|\mathcal{M}| = 2^{n(H(X|U)+\delta)}$ and

$$P_{XU}^{(n)}(\mathcal{A}^c) \leq \epsilon \quad (49)$$

for sufficiently large n , where

$$\mathcal{A} := \{(x^n, u^n) \in \mathcal{X}^n \times \mathcal{U}^n : \mathcal{D}(\varphi(x^n), u^n) = x^n\} \quad (50)$$

is the set of correctly decodable pairs.

Now, suppose that Alice and Bob use the same code for the simulated distribution $Q_{X\tilde{U}}^n$. Then, by the definition of the variational distance and (48), we have

$$Q_{X\tilde{U}}^n(\mathcal{A}^c) \leq P_{XU}^{(n)}(\mathcal{A}^c) + \epsilon. \quad (51)$$

Thus, if we can find a good code for $P_{XU}^{(n)}$, we can also use that code for $Q_{X\tilde{U}}^n$ for sufficiently large n .

Derandomization. The standard derandomization technique works here. Since the distribution $Q_{X\tilde{U}}^n = \frac{1}{|\mathcal{K}|} \sum_{k \in \mathcal{K}} Q_{X\tilde{U}|k}^n$, and

$$\sum_k \frac{1}{|\mathcal{K}|} Q_{X\tilde{U}|K=k}^n(\mathcal{A}^c) = Q_{X\tilde{U}}^n(\mathcal{A}^c) \leq P_{XU}(\mathcal{A}^c) + \epsilon. \quad (52)$$

Thus, there exists one $k \in \mathcal{K}$ so that $Q_{X\tilde{U}|k}^n(\mathcal{A}_n^c)$ is small. ■

III. CONCLUSION AND DISCUSSION

We considered the problem of compression of a classical source with a quantum helper. We completely characterised its rate region and showed that the capacity formula does not require regularisation, which is not common in the quantum setting. While the expressions for the rate region are similar to the classical result in [3], [4], [21], it requires very different proof technique. To prove the achievability, we employed a powerful theorem, measurement compression theorem [22], that can decompose quantum measurement. A similar approach was recently applied to derive a non-asymptotic bound on the classical helper problem [24].

This work brings more questions than answered. As we have pointed out, source coding with a helper was never considered in the quantum regime before ours. Our work can be served as the first step to the more general (fully quantum) setting; namely, quantum source coding with a quantum helper. Currently, it is completely unknown how to quantify the distinction between side information and a quantum helper. We believe that resolving this question will sharpen our understanding of a quantum source.

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